Polynomial Degree and End Behavior (1.5+1.6) Homework

1. Let $f(x) = -3x^5 + 11x^2 + 2x - 7$. Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

2. Let $g(x) = 2x^4 - x^3 + 18$. Determine the end behavior of g as x decreases without bound. Express your answer using the mathematical notation of a limit.

3. Let $h(x) = -2x^5 + 8x$. Determine the end behavior of h as x increases without bound. Express your answer using the mathematical notation of a limit.

4. $h(x)$ is a polynomial function. Given that $\lim_{x\to-\infty} h(x) = -\infty$ and $\lim_{x\to\infty} h(x) = \infty$, which one of the following is possible?
   A) The leading coefficient of $h(x)$ is 10 and the degree of $h(x)$ is 5
   B) The leading coefficient of $h(x)$ is -6 and the degree of $h(x)$ is 5
   C) The leading coefficient of $h(x)$ is -5 and the degree of $h(x)$ is 4
   D) The leading coefficient of $h(x)$ is 4 and the degree of $h(x)$ is 6

5. Let $f(x) = (3x^2 + 4)(3 - 2x)^3(2x^2 - 4x)^3$. Determine the end behavior of f as x decreases without bound. Express your answer using the mathematical notation of a limit.

6. Let $g(x) = (3 - x^2)(5x + 3)^4(x^3 - 5)(-2x + 3)$. Determine the end behavior of g as x increases without bound. Express your answer using the mathematical notation of a limit.

7. Below are values for $g(x)$, a cubic polynomial. Find the value of $g(11)$.

x	-4	-1	2	5	8	11	14
$g(x)$	-10	1	2	-1	-2		26

8. What is the least possible degree of a polynomial that would fit the data below?

x	-3	-2	-1	0	1	2	3
$f(x)$	8	37	40	41	40	13	-88

9. What is the least possible degree of a polynomial that would fit the data below?

x	-6	-4	-2	0	2	4	6	8
$f(x)$	-6	-1	0	-1	-2	-1	4	15

10. Polly analyzes the values of $h(x)$ given below and makes the conclusion "h is best modeled by a quadratic function since consecutive equally spaced input values correspond with a constant increase in the second differences of the output values." Which of the following are wrong with Polly's reasoning? **Select all that apply**.

x	1	2	4	8	16	32	64
$h(x)$	14	5	0	-1	2	9	20

A) Nothing is wrong with her reasoning. She is a treasure, and how dare you impugn her genius?!
B) Polly implied the second differences are increasing. They are not. They are constant.
C) She said "consecutive equally spaced input values," but the input values are not equally spaced.
D) $h(x)$ is cubic (degree 3), not quadratic (degree 2).
E) Polly should have included numbers in her justification to prove she looked at the table rather than just wrote out a sentence her teacher told her to memorize.

11. The table below gives values for the polynomial $f(x)$ at selected values of x.

x	-6	-4	-2	0	2	4	6
$f(x)$	11	3	-2	-4	-3	1	8

$f(x)$ is of degree 2. For each of the following statements, decide if the statement is a correct and valid justification for why the least possible degree of $f(x)$ is 2.

Statement 1: Because second differences in the output values are constant $(+3)$ over equal-length input value intervals, the least possible degree of f is 2.
Statement 2: Over consecutive equal-length input-value intervals, the average rates of change of $f(x)$ have a constant difference $(+1.5)$. Therefore, the least possible degree of f is 2.
Statement 3: Because second differences in the output values increase by a constant amount $(+3)$ over consecutive equal-length input-value intervals, the least possible degree of f is 2.
Statement 4: Because first differences in the output values increase by a constant amount $(+3)$ over consecutive equal-length input-value intervals, the least possible degree of f is 2.

Each of the following questions describes a polynomial function. However, each scenario is impossible. Give a reason why the proposed polynomial is impossible.

12. $f(1) = 0$, $f(4) = 3$, the absolute maximum value of $f(x)$ is 5, f is a fifth degree polynomial

13. $\lim_{x\to-\infty} f(x) = -\infty$, $\lim_{x\to\infty} f(x) = \infty$, the leading coefficient of $f(x)$ is positive, $y = f(x)$ has no x-intercepts

14. f is quartic (degree = 4), $y = f(x)$ has no x-intercepts, f has no relative extrema, $\lim_{x\to\infty} f(x) = \infty$

15. $f(0) = 5$, $f(4) = 8$, $f(10) = 12$, f is concave up over $(-\infty, \infty)$

16. $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x)$, $f(2) = 5$ is a relative minimum, $f(x)$ is concave up over $(-1,5)$, f is of degree 5.

Polynomial Degree and End Behavior (1.5+1.6) HW Answers

1. $\lim_{x\to\infty} f(x) = -\infty$ and $\lim_{x\to-\infty} f(x) = \infty$
2. $\lim_{x\to-\infty} g(x) = \infty$ (note: there aren't other ways to write this. This is the one and only correct answer. You need the negative under the word $\lim$. You need the $g(x)$. If you wrote $f(x)$, you're close, but close only counts in archery and slow dances)
3. $\lim_{x\to\infty} h(x) = -\infty$ (note: there aren't other ways to write this. This is the one and only correct answer. You need the word $\lim$. You need the $x \to \infty$. You need the $h(x)$. You need the equals sign. And you need to $-\infty$. Were you almost correct but you messed up one little part. There is no partial credit here. "Almost correct" is nice, but almost only counts in shuffleboard and meteor misses.)
4. A
5. As x decreases without bound, $f(x)$ increases without bound. In other words, $\lim_{x\to-\infty} f(x) = \infty$.
6. As x increases without bound, $f(x)$ increases without bound. In other words, $\lim_{x\to\infty} f(x) = \infty$.
7. $g(11) = 5$
8. A fourth degree polynomial (aka a quartic polynomial)
9. A third degree polynomial (aka a cubic polynomial)
10. B, C, and E. Three errors? Get your act together Polly.
11.

x	-6	-4	-2	0	2	4	6
$f(x)$	11	3	-2	-4	-3	1	8

First Differences: -8 -5 -2 $+1$ $+4$ $+7$

Second Differences: $+3$ $+3$ $+3$ $+3$ $+3$

Statement 1: Valid. The second differences are constantly +3 over these input-value intervals that are all equally wide.

Statement 2: Valid. Average rate of change is difference in output value divided by difference input value. If we divide all the first differences by 2 we see the average rates of change over consecutive equal-width input-value intervals are $-4, -2.5, -1, 0.5, 2, 3.5$. These do indeed have a constant difference. Looking at differences in average rates of change is kind of like looking at second differences. A constant difference in average rate of change is kind of like a constant second difference in output value.

Statement 3: NOT VALID. It is false to say the second differences are increasing by a constant amount. The second differences aren't increasing at all. They are constantly $+3$.

Statement 4: Valid. The first differences are increasing by a constant amount. That's just another way of saying the second differences are constant.

12. A fifth-degree polynomial (or any odd degree polynomial) cannot have an absolute maximum. When the left and right end behavior of a polynomial do not match, the range of the polynomial must be $(-\infty, \infty)$. Therefore, there cannot be one output value greater than or equal to all other output values.
13. When the left and right end behavior of a polynomial do not match, the range of the polynomial must be $(-\infty, \infty)$. At some point, there must be an output value of zero. That corresponds with an x-intercept. In other words, since $\lim_{x\to-\infty} f(x) = -\infty$ there are certainly negative output values for the function. Since $\lim_{x\to\infty} f(x) = \infty$ there are certainly positive output values for the function. Since polynomials are continuous, somewhere between a point with a negative output value and a point with a positive output value is a point with an output value of zero.
14. All even degree polynomials have either an absolute maximum or an absolute minimum value. Whichever it is, that point must also be a relative extremum.

15. The average rate of change over the interval $[0,4]$ is $\frac{f(4)-f(0)}{4-0} = \frac{3}{4}$. The average rate of change over the interval $[4,10]$ is $\frac{f(10)-f(4)}{10-4} = \frac{2}{3}$. Since $\frac{2}{3} < \frac{3}{4}$, the rate of change is not increasing. The function is not concave up.
16. For a fifth-degree polynomial (as with any odd degree polynomial), the right end behavior and the left end behavior will differ. $\lim_{x\to\infty} f(x) \neq \lim_{x\to-\infty} f(x)$.